Homogeneous (α, k)-polynomial solutions of the fractional Riesz system in hyperbolic space*

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Abstract

In this paper we study the fractional analogous of the Laplace-Beltrami equation and the Riesz system studied previously by H. Leutwiler , in $\mathbb{R}^3$. In both cases we replace the integer derivatives by Caputo fractional derivatives of order $0 < \alpha < 1$. We characterize the space of solutions of the fractional Laplace-Beltrami equation, and we calculate its dimension. We establish relations between the solutions of the fractional Laplace-Beltrami equation and the solutions of the fractional Riesz system. Some examples of the polynomial solutions will be presented. Moreover, the behaviour of the obtained results when $\alpha = 1$ is presented, and a final remark about the consideration of Riemann-Liouville fractional derivatives instead of Caputo fractional derivatives is made.

Keywords: Hypermonogenic functions; Laplace-Beltrami fractional differential operator; Caputo fractional derivative; Hyperbolic fractional Riesz system; Hyperbolic.

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1 Introduction

One of the possible extensions of the theory of classical complex analysis to higher dimensions using geometric algebras is the theory of hypermonogenic functions based on the hyperbolic model. The advantage of hypermonogenic functions is that the positive and negative powers of hypercomplex variables are included into the theory, which is not in the monogenic case. Hence, the elementary functions can be defined similarly as in the classical complex case. In [7]- [9], H. Leutwiler and S.L. Eriksson introduced hypermonogenic and $\kappa-$hypermonogenic functions, and studied some of their properties in Clifford analysis. Hypermonogenic functions are generalizations of the monogenic functions and the $\kappa-$monogenic functions are extensions of the hypermonogenic functions. When $\kappa = n - 1$, a $\kappa-$hypermonogenic function is a hypermonogenic function, and when $\kappa = 0$, a $\kappa-$hypermonogenic function is a monogenic function. In 1992 H. Leutwiler published two papers [11,12] where he studied the connections between the solutions of the so-called hyperbolic Riesz system and the solutions of the Laplace-Beltrami equation.

Recently, the interest in fractional calculus verified a substancial increment. Among all the possible connections that this topic can establish with several areas of mathematics, there is a lot of interest in the study of

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ordinary and partial fractional differential equations regarding their mathematical aspects and their applications in diverse areas such as physics, chemistry, engineering, optics or quantum mechanics (see [2, 4]).

The aim of this paper is to study, in $\mathbb{R}^3$, the analogous of the hyperbolic Riesz system and the Laplace-Beltrami equation for the case where the integer derivatives are replaced by the Caputo fractional derivatives. For the sake of simplicity we restrict ourselves to the three dimensional case, however the results can be generalized for an arbitrary dimension. We point out that there are previous works where fractional derivatives are considered in the context of Clifford analysis (see [1, 3, 17]), however they did not consider the hyperbolic model. This works corresponds, as far as the authors are aware, to the first connection between the theory of hypermogenic functions and fractional calculus.

The structure of the paper reads as follows: in the Preliminary section we recall some basic facts about hypermonogenic functions and fractional calculus, which are necessary to the development of this work. In Section 3 we study the (α, k)-harmonic polynomial solutions of the fractional Laplace-Beltrami equation, and we characterize the elements of the space of (α, k)-homogeneous polynomials. Moreover, we study the dimension of this space. In the end of the section we present some examples of solutions of the fractional Laplace-Beltrami equation. In the Section 4, we study the fractional version of the hyperbolic Riesz system. We establish relations between the solutions of this system and the solutions obtained in Section 3 for the fractional Laplace-Beltrami equation. We also study the dimension of the space of solutions of the hyperbolic fractional Riesz system. We end the section presenting some examples of polynomial solutions of the fractions Ryes system. In Sections 3 and 4, the behaviour of our results when $\alpha = 1$ and the connections of this particular case with the work of H. Leutwiler will be studied. In the end of this paper we present a final remark about the possibility of consider Riemann-Liouville fractional derivatives instead of Caputo fractional derivatives.

2 Preliminaries

Let us recall some standard facts from hyperbolic geometry, to make our point of view clear (for more details see [5–13]). The Poincaré upper half space is a Riemannian manifold $(\mathbb{R}^3_+, ds^2)$ with the metric

$$ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2}{x_0^2},$$

and the Laplace-Beltrami operator on the manifold is

$$\Delta_{lb} f = x_0^2 \Delta f - x_0 \partial_{x_0} f,$$

which is also called the hyperbolic Laplace operator. We point out that our operator is a special case of a Weinstein operator (see e.g. [6]). Distance in the hyperbolic upper half space may be computed as follows (see [13]).

Lemma 2.1 *The hyperbolic distance $d_h(x, a)$ between the points x and a in $\mathbb{R}^3_+$ is*

$$d_h(x, a) \ = \operatorname{arcosh}\left(\lambda(x, a)\right) \ = \ln\left(\lambda(x, a) + \sqrt{\lambda(x, a)^2 - 1}\right),$$

where

$$\lambda(x, a) \ = \frac{|x - a|^2 + |x - \hat{a}|^2}{4x_2 a_2} \ = \frac{|x - a|^2}{2x_2 a_2} + 1 \tag{1}$$

and $|x - a|$ is the usual Euclidean distance in $\mathbb{R}^3$ between the points a and x.

Now, we recall the following important relation between the Euclidean and hyperbolic balls.

Proposition 2.2 *(cf. [13]) The hyperbolic ball $B_h(a, r_h)$ in $\mathbb{R}^3_+$ with the hyperbolic center $a = a_0 + a_1 e_1 + a_2 e_2$ and the radius r_h is the same as the Euclidean ball with the Euclidean center*

$$c_a(r_h) = a_0 + a_1 e_1 + a_2 \cosh r_h e_2$$

and the Euclidean radius $r_e = a_2 \sinh r_h$.

Another operators on $\mathbb{R}^3_+$ are the exterior derivative d and its adjoint d^* (see [15]). It is easy to see that if we consider a one form $\omega^1 = u_0 dx_0 + u_1 dx_1 + u_2 dx_2$ in the upper half space, it satisfies $d\omega^1 = d^*\omega^1 = 0$ if and only if the component functions satisfies the so-called hyperbolic Riesz system

$$\begin{cases} x_0 \left(\dfrac{\partial u_0}{\partial x_0} + \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} \right) - u_0 = 0 \\ \dfrac{\partial u_k}{\partial x_j} = \dfrac{\partial u_j}{\partial x_k}, \quad \text{with } k < j, \quad k, j = 0, 1, 2 \end{cases}. \tag{2}$$

This system is an analogy to the classical Cauchy-Riemann system on a plane and the Stein-Weiss system in higher dimensions, which characterize harmonic 1-forms with respect to Euclidean metric. The Laplace-Beltrami equation and the hyperbolic Riesz system alike may define not only $\mathbb{R}^3_+$ but also in the whole space $\mathbb{R}^3$. In this case, if we have a differentiable function h, defined on the whole $\mathbb{R}^3$ and satisfies $\Delta_{lb} h = 0$, it is a standard trick that the components of its gradient $(u_0, u_1, u_2) = \nabla h$ satisfies the hyperbolic Riesz system. Conversely if u_0, u_1 and u_2 is a solution of the hyperbolic Riesz system, we may find a scalar potential h, satisfying the equation $\Delta_{lb} h = 0$.

Now we recall some basic facts about fractional calculus and we fix some of the notations used in the paper. Let ${}^C\partial^\alpha_w$ denote the Caputo fractional derivative of order $\alpha > 0$ with respect to the variable w (see [4])

$$\left({}^C\partial^\alpha_w f \right)(w) \ = \left(I^{n-\alpha}_w D^n f \right)(w) \ = \frac{1}{\Gamma(n-\alpha)} \int_0^w \frac{f^{(n)}(t)}{(w-t)^{\alpha-n+1}} \, dt, \tag{3}$$

where $w > 0$, $n = [\alpha] + 1$ with $[\alpha]$ denoting the integer part of α, and I^α_w denotes the Riemann-Liouville fractional integral of order α with respect to the variable w, i.e., (see [4])

$$\left(I^\alpha_w f \right)(w) \ = \ \frac{1}{\Gamma(\alpha)} \int_0^w \frac{f(t)}{(w-t)^{1-\alpha}} \, dt, \qquad w > 0, \quad \alpha > 0.$$

For $\alpha = n \in \mathbb{N}$, the Caputo fractional derivative coincides with the standard derivative of order n. Moreover, in (3) the function $f(w)$ belongs to $AC^n[0, b]$. $AC^n([0, b])$ denotes the class of functions f, which are continuously differentiable on the segment $[0, b]$ up to order $n - 1$ and $f^{(n-1)}$ is absolutely continuous on $[0, b]$. When $\alpha \notin \mathbb{N}_0$, the Caputo fractional derivative ${}^C\partial^\alpha_w$ corresponds to the left inverse of the Riemann-Liouville fractional integral I^α_w, i.e, $\left({}^C\partial^\alpha_w \, I^\alpha_w \, f \right)(w) = f(w)$, with $f(w) \in C[0, b]$ (see [4]). The following result characterizes the composition of the fractional integral operator I^α_w with the fractional derivative ${}^C\partial^\alpha_w$.

Lemma 2.3 *(cf. [4]) Let $\alpha \in \mathbb{R}^+$ and $n = [\alpha] + 1$. If $f(w) \in AC^n[0, b]$, then*

$$\left(I^\alpha_w {}^C\partial^\alpha_w f \right)(w) \ = \ f(w) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \, w^k.$$

In particular, if $0 < \alpha < 1$ and $f(w) \in AC[0, b]$, then

$$\left(I^\alpha_w {}^C\partial^\alpha_w f \right)(w) \ = \ f(w) - f(0).$$

For more details about fractional calculus and its basics definitions see [4], for example.

3 (α, k)-harmonic polynomial solutions for Laplace-Beltrami operator

In this section we study the null polynomial solutions of the fractional Laplace-Beltrami operator, i.e., we study the existence of polynomial solutions of the equation

$$\Delta^\alpha_{lb} h(x_0, x_1, x_2) \ = \left(x^\alpha_0 \Delta^\alpha - {}^C\partial^\alpha_{x_0} \right) h(x_0, x_1, x_2) \ = 0, \tag{4}$$

where $\Delta^\alpha = \sum_{i=0}^2 {}^C\partial^\alpha_{x_i} \, {}^C\partial^\alpha_{x_i}$ and $0 < \alpha \leq 1$. If $\mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)}$ is the space of (α, k)-homogeneous polynomials, we define

$$\mathcal{H}^{(\alpha,k)} = \left\{ P^{(\alpha,k)} \in \mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)} : \Delta^\alpha_{lb} P^{(\alpha,k)}(x_0, x_1, x_2) = 0 \right\},$$

i.e., the space of (α, k)-harmonic polynomial solutions of the fractional Laplace-Beltrami operator. Denoting $\Delta^\alpha_{(x_0,x_1)} = {}^C\partial^\alpha_{x_1}\,{}^C\partial^\alpha_{x_1} + {}^C\partial^\alpha_{x_0}\,{}^C\partial^\alpha_{x_0}$ and $H^\alpha = x_0^\alpha\,{}^C\partial^\alpha_{x_0}\,{}^C\partial^\alpha_{x_0} - {}^C\partial^\alpha_{x_0}$, we have that $\Delta^\alpha_{lb} = x_0^\alpha \Delta^\alpha_{(x_1,x_2)} + H^\alpha$. We consider (α, k)-homogeneous polynomials of the form

$$\mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)} = \bigoplus_{j=0}^{k} \mathcal{P}^{(\alpha,j)}_{(x_1,x_2)} \otimes \mathcal{P}^{(\alpha,k-j)}_{(x_0)},$$

i.e., every polynomial is in the form

$$P^{(\alpha,k)}(x_0, x_1, x_2) = \sum_{j=0}^{k} Q_j^{(\alpha,k)}(x_1, x_2)\, x_0^{\alpha(k-j)}, \tag{5}$$

where $Q_j^{(\alpha,k)}$ is a (α, k)-homogeneous polynomial. These polynomials correspond bijectively to the $k+1$-tuples of (α, k)-homogeneous polynomials, which are denoted by

$$\overrightarrow{P}^{(\alpha,k)}(x_1, x_2) = \left(Q_0^{(\alpha,k)}(x_1, x_2),\, Q_1^{(\alpha,k)}(x_1, x_2),\, \ldots,\, Q_k^{(\alpha,k)}(x_1, x_2)\right).$$

In the next theorem we characterize the elements of $\mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)}$.

Theorem 3.1 *Let $0 < \alpha \leq 1$. The equality $\Delta^\alpha_{lb} P^{(\alpha,k)} = 0$, with $P^{(\alpha,k)} \in \mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)}$ given by (5), holds if and only if*

$$\begin{cases} Q_0^{(\alpha,k)}(x_1, x_2) \in \mathcal{P}^{(\alpha,0)}_{(x_1,x_2)} \\ Q_1^{(\alpha,k)}(x_1, x_2) \in \mathcal{P}^{(\alpha,1)}_{(x_1,x_2)} \\ \Delta^\alpha_{(x_1,x_2)} Q_{j+1}^{(\alpha,k)}(x_1, x_2) + \varphi(k-j-1)\,\varphi(k-j)\, Q_{j-1}^{(\alpha,k)}(x_1, x_2) = 0 \\ Q_{k-1}^{(\alpha,k)}(x_1, x_2) = 0 \end{cases}.$$

where $j = 1, ..., k-1$.

Proof: First we observe that for $f(w) = w^k$, with $0 < \alpha < 1$ and $k \in \mathbb{N}$, we have that

$$\left({}^C\partial^\alpha_w f\right)(w) \;=\; \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}\, w^{k-\alpha} \;=: \varphi(k)\, w^{k-\alpha}. \tag{6}$$

Moreover, when $k = 0$ it is immediate that $\left({}^C\partial^\alpha_w f\right)(w) = 0$. Making straightforward calculations we arrive to

$$\Delta^\alpha_{lb} P^{(\alpha,k)}(x_0, x_1, x_2) \;=\; \sum_{j=0}^{k} \Delta^\alpha_{(x_1,x_2)} Q_j^{(\alpha,k)}(x_1, x_2)\, x_0^{\alpha(k-j+1)} + \sum_{j=0}^{k} Q_j^{(\alpha,k)}(x_1, x_2)\, H^\alpha x_0^{\alpha(k-j)}. \tag{7}$$

Furthermore, we have

$$H^\alpha x_0^{\alpha(k-j)} = \varphi(\alpha(k-j))\, \left(\varphi(\alpha(k-j-1)) - 1\right) x_0^{\alpha(k-j-1)}, \tag{8}$$

with $j = 0, \ldots, k$. In particular when $j = k$ we have $H x_0^0 = 0$, and

$$\Delta^\alpha_{(x_1,x_2)} Q_0^{(\alpha,k)}(x_1, x_2) = \Delta^\alpha_{(x_1,x_2)} Q_1^{(\alpha,k)}(x_1, x_2) \;= 0. \tag{9}$$

Hence immersing (8) and (9) into (7), and changing the order of summation, we get

$$\begin{aligned} &\Delta^\alpha_{lb} P^{(\alpha,k)}(x_0, x_1, x_2) \\ &= \sum_{j=2}^{k} \Delta^\alpha_{(x_1,x_2)} Q_j^{(\alpha,k)}(x_1, x_2)\, x_0^{\alpha(k-j+1)} + \sum_{j=0}^{k-1} \varphi(\alpha(k-j))\, \left(\varphi(\alpha(k-j-1)) - 1\right)\, Q_j^{(\alpha,k)}(x_1, x_2)\, x_0^{\alpha(k-j-1)} \\ &= \sum_{j=1}^{k-1} \Delta^\alpha_{(x_1,x_2)} Q_{j+1}^{(\alpha,k)}(x_1, x_2)\, x_0^{\alpha(k-j)} + \sum_{j=1}^{k} \varphi(\alpha(k-j+1))\, \left(\varphi(\alpha(k-j)) - 1\right)\, Q_{j-1}^{(\alpha,k)}(x_1, x_2)\, x_0^{\alpha(k-j)}, \end{aligned} \tag{10}$$

Ir order to (10) be equal to zero, we conclude that $Q_{k-1}^{(\alpha,k)}(x_1, x_2) = 0$ and

$$0 \;=\; \Delta^\alpha_{(x_1,x_2)} Q_{j+1}^{(\alpha,k)}(x_1, x_2) + \varphi(\alpha(k-j+1))\, \left(\varphi(\alpha(k-j)) - 1\right)\, Q_{j-1}^{(\alpha,k)}(x_1, x_2) \tag{11}$$

for $j = 1, ..., k-1$.

$\blacksquare$

Remark 3.2 *Note that in Euclidean case, i.e. when $\alpha = 1$, we obtain an extra condition $\Delta_{(x_1,x_2)} Q_k^{(1,k)}(x_1,x_2) = 0$.*

We now study the dimension of the space $\mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)}$. The authors will like to point out that in the calculus of the dimensions the fractional parameter α has not influence. This fact implies the result concerning to the dimension on the spaces are equal to the correspondent ones in the case of integer derivatives. Recalling that the dimension of the k-homogeneous polynomials with n variables is $\binom{n+k-1}{n-1}$ (see [16]), we have

$$\dim\left(\mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)}\right) = \frac{(k+1)(k+2)}{2}, \qquad \dim\left(\mathcal{P}^{(\alpha,j)}_{(x_1,x_2)}\right) = j+1,$$
$$\dim\left(\mathcal{P}^{(\alpha,k-j)}_{(x_0)}\right) = 1.$$

We have also the following decomposition

$$\mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)} = \bigoplus_{j=0}^{k} \mathcal{P}^{(\alpha,j)}_{(x_1,x_2)} \otimes \mathcal{P}^{(\alpha,k-j)}_{(x_0)},$$

which implies that

$$\dim\left(\mathcal{P}^{(\alpha,k)}_{(x_0,x_1,x_2)}\right) = \sum_{j=0}^{k} \dim\left(\mathcal{P}^{(\alpha,j)}_{(x_1,x_2)}\right).$$

Taking into account [16] we have that the dimension of the space of (α, j)-homogeneous harmonic polynomials in the Euclidean space

$$\mathcal{H}^{(\alpha,k)}\left(\mathbb{R}^2\right) = \left\{P^{(\alpha,k)} \in \mathcal{P}^{(\alpha,k)}_{(x_1,x_2)} : \Delta^{\alpha}_{(x_1,x_2)} P^{(\alpha,k)} = 0\right\}$$

is given by

$$\dim\left(\mathcal{H}^{(\alpha,k)}\left(\mathbb{R}^2\right)\right) = 2, \qquad k \geq 2.$$

Since $Q^{(\alpha,k)}_{k-1} = 0$ we obtain the following result from the previous relations:

Theorem 3.3 *Let $0 < \alpha \leq 1$. If k is odd then $Q^{(\alpha,k)}_{k-1} = Q^{(\alpha,k)}_{k-3} = \ldots = Q^{(\alpha,k)}_{2} = Q^{(\alpha,k)}_{0} = 0$, i.e.,*

$$\overrightarrow{P}^{(\alpha,k)}(x_1,x_2) = \left(0, Q^{(\alpha,k)}_1(x_1,x_2), 0, Q^{(\alpha,k)}_3(x_1,x_2), \ldots, Q^{(\alpha,k)}_{k-2}(x_1,x_2), 0, Q^{(\alpha,k)}_k(x_1,x_2)\right),$$

and if k is even then $Q^{(\alpha,k)}_{k-1} = Q^{(\alpha,k)}_{k-3} = \ldots = Q^{(\alpha,k)}_{3} = Q^{(\alpha,k)}_{1} = 0$, i.e.,

$$\overrightarrow{P}^{(\alpha,k)}(x_1,x_2) = \left(Q^{(\alpha,k)}_0(x_1,x_2), 0, Q^{(\alpha,k)}_2(x_1,x_2), \ldots, Q^{(\alpha,k)}_{k-2}(x_1,x_2), 0, Q^{(\alpha,k)}_k(x_1,x_2)\right).$$

Moreover from (11) we can establish the following recurrence relations for computations:

- If k is odd

$$Q^{(\alpha,k)}_1 \longrightarrow Q^{(\alpha,k)}_3 \longrightarrow Q^{(\alpha,k)}_5 \longrightarrow \ldots \longrightarrow Q^{(\alpha,k)}_k. \tag{12}$$

- If k is even

$$Q^{(\alpha,k)}_0 \longrightarrow Q^{(\alpha,k)}_2 \longrightarrow Q^{(\alpha,k)}_2 \longrightarrow \ldots \longrightarrow Q^{(\alpha,k)}_k. \tag{13}$$

Taking into account the properties of the Caputo fractional derivatives described in [4] we have that the operator $\Delta^{\alpha}_{(x_1,x_2)}$ is a surjection. Moreover, using the isomorphism theorem (see [14]), we can prove the following result:

Theorem 3.4 *Let $0 < \alpha \leq 1$. The operator $\Delta^{\alpha}_{(x_1,x_2)} : \mathcal{P}^{(\alpha,k)}_{(x_1,x_2)}/\mathcal{H}(\mathbb{R}^2)^{(\alpha,k)} \to \mathcal{P}^{(\alpha,k-2)}_{(x_1,x_2)}$ is isomorphism.*

From the previous theorem we obtain our result related with dimension of the space $\mathcal{H}^k_\alpha$.

Theorem 3.5 *Let $0 < \alpha \leq 1$. The dimension of the space $\mathcal{H}_{\alpha}^{k}$, with $0 < \alpha < 1$, is given by:*

$$\dim\left(\mathcal{H}^{(\alpha,k)}\right) = \begin{cases} k+1, & (k \text{ odd}) \\ \dfrac{k}{2}+1, & (k \text{ even}) \end{cases}.$$

Proof: Assume that $k = 2j+1$, i.e., k is odd. It is immediate the space of $(\alpha, 1)$-harmonic functions has dimension 2. Moreover, using the recurrence relation (12), we can construct a pair of harmonic polynomials for every $\mathcal{P}^{3}_{(x_1,x_2)}$, $\mathcal{P}^{5}_{(x_1,x_2)}$. We have

$$\dim\left(\mathcal{H}^{(\alpha,k)}\right) = \sum_{i=0}^{\frac{k-1}{2}} 2 = k+1$$

Now let us consider that $k = 2j$, i.e., k is even. In this case the dimension of space of $(\alpha, 0)$-harmonic functions is 1. Proceeding in a similar way, as it was done for the odd case, we conclude that

$$\dim\left(\mathcal{H}^{(\alpha,k)}\right) = \sum_{i=0}^{\frac{k}{2}} 1 = \frac{k+2}{2},$$

which completes the proof.

■

Now we present some examples of polynomials of the form

$$P^{(\alpha,k)}(x_0,x_1,x_2) = \sum_{\substack{a_0,a_1,a_2=0\\ a_0+a_1+a_2=k}}^{k} \beta_{a_0,a_1,a_2}\, x_0^{a_0\alpha}\, x_1^{a_1\alpha}\, x_2^{a_2\alpha}$$

belonging to $\mathcal{H}^{(\alpha,k)}$, for some particular values of k and $0 < \alpha < 1$ arbitrary.

Example 3.6 *For $k = 1$ we have the following element of $\mathcal{H}^{(\alpha,1)}$*

$$P^{(\alpha,1)}(x_0,x_1,x_2) = \beta_{(0,1,0)}\, x_1^{\alpha} + \beta_{(0,0,1)}\, x_2^{\alpha}. \tag{14}$$

For $k = 2$ we have the following element of $\mathcal{H}^{(\alpha,2)}$

$$P^{(\alpha,2)}(x_0,x_1,x_2) = \beta_{(2,0,0)}\, x_0^{2\alpha} - \left(\frac{\varphi(\alpha)-1}{\varphi(\alpha)}\,\beta_{(2,0,0)} + \beta_{(0,0,2)}\right) x_1^{2\alpha} + \beta_{(0,1,1)}\, x_1^{\alpha}\, x_2^{\alpha} + \beta_{(0,0,2)}\, x_2^{2\alpha}. \tag{15}$$

For $k = 3$ we have the following element of $\mathcal{H}^{(\alpha,3)}$

$$\begin{aligned} P^{(\alpha,3)}(x_0,x_1,x_2) &= \beta_{(2,1,0)}\, x_0^{2\alpha}\, x_1^{\alpha} + \beta_{(2,0,1)}\, x_0^{2\alpha}\, x_2^{\alpha} - \left(\frac{\varphi(\alpha)}{\varphi(3\alpha)}\,\beta_{(0,1,2)} + \frac{\varphi(\alpha)-1}{\varphi(3\alpha)}\,\beta_{(2,1,0)}\right) x_1^{3\alpha} \\ &\quad - \left(\frac{\varphi(3\alpha)}{\varphi(\alpha)}\,\beta_{(0,0,3)} + \frac{\varphi(\alpha)-1}{\varphi(\alpha)}\,\beta_{(2,0,1)}\right) x_1^{2\alpha}\, x_2^{\alpha} + \beta_{(0,1,2)}\, x_1^{\alpha}\, x_2^{2\alpha} + \beta_{(0,0,3)}\, x_2^{3\alpha}. \end{aligned} \tag{16}$$

For $k = 4$ we have the following element of $\mathcal{H}^{(\alpha,4)}$

$$\begin{aligned} P^{(\alpha,4)}(x_0,x_1,x_2) &= -\frac{\varphi(2\alpha)\,\varphi(\alpha)}{\varphi(4\alpha)\,(\varphi(3\alpha)-1)}\left(\beta_{(2,2,0)} + \beta_{(2,0,2)}\right) x_0^{4\alpha} + \beta_{(0,0,4)}\, x_2^{4\alpha} + \beta_{(2,2,0)}\, x_0^{2\alpha}\, x_1^{2\alpha} \\ &\quad - \left(\beta_{(0,1,3)} + \frac{\varphi(\alpha)-1}{\varphi(3\alpha)}\beta_{(2,1,1)}\right) x_1^{3\alpha}\, x_2^{\alpha} + \beta_{(2,1,1)}\, x_0^{2\alpha}\, x_1^{\alpha}\, x_2^{\alpha} + \beta_{(2,0,2)}\, x_0^{2\alpha}\, x_2^{2\alpha} \\ &\quad + \beta_{(0,1,3)}\, x_1^{\alpha}\, x_2^{3\alpha} - \left(\frac{\varphi(4\alpha)\,\varphi(3\alpha)}{\varphi(2\alpha)\,\varphi(\alpha)}\,\beta_{(0,0,4)} + \frac{\varphi(\alpha)-1}{\varphi(\alpha)}\,\beta_{(2,0,2)}\right) x_1^{2\alpha}\, x_2^{2\alpha} \\ &\quad + \left(\beta_{(0,0,4)} + \frac{\varphi(2\alpha)\,(\varphi(\alpha)-1)}{\varphi(4\alpha)\,\varphi(3\alpha)}\left(\beta_{(2,0,2)} - \beta_{(2,2,0)}\right)\right) x_1^{4\alpha}. \end{aligned} \tag{17}$$

Remark 3.7 *If we consider $\alpha = 1$ in the polynomials presented in the previous example, we obtain the polynomial solutions of $\Delta_{lb} f = 0$, which were introduced by H. Leutwiler. More precisely, we have*

$$\begin{aligned}
P^{(1,1)}(x_0,x_1,x_2) &= \beta_{(0,1,0)}\, x_1 + \beta_{(0,0,1)}\, x_2,\\
P^{(1,2)}(x_0,x_1,x_2) &= \beta_{(2,0,0)}\, x_0^2 - \beta_{(0,0,2)}\, x_1^2 + \beta_{(0,1,1)}\, x_1\, x_2 + \beta_{(0,0,2)}\, x_2^2,\\
P^{(1,3)}(x_0,x_1,x_2) &= \beta_{(2,1,0)}\, x_0^2\, x_1 + \beta_{(2,0,1)}\, x_0^2\, x_2 - \frac{1}{3}\beta_{(0,1,2)}\, x_1^3 - 3\beta_{(0,0,3)}\, x_1^2\, x_2 + \beta_{(0,1,2)}\, x_1\, x_2^2 + \beta_{(0,0,3)}\, x_2^3,\\
P^{(1,4)}(x_0,x_1,x_2) &= -\frac{1}{4}\left(\beta_{(3,0,1)} + \beta_{(2,0,2)}\right) x_0^4 + \beta_{(2,2,0)}\, x_0^2\, x_1^2 + \beta_{(2,1,1)}\, x_0^2\, x_1\, x_2 + \beta_{(2,0,2)}\, x_0^2\, x_2^2 + \beta_{(0,0,4)}\, x_1^4\\
&\quad -\beta_{(0,1,3)}\, x_1^3\, x_2 + \beta_{(0,1,3)}\, x_1\, x_2^3 + \beta_{(0,0,4)}\, x_2^4 - 6\beta_{(0,0,4)}\, x_1^2\, x_2^2.
\end{aligned}$$

4 (α, k)-homogeneous solutions the hyperbolic fractional Riesz systems

The aim of this section is to study the fractional analogous of the hyperbolic Riesz system (2), which we will call hyperbolic fractional Riesz system

$$\left\{\begin{array}{l}
x_0^\alpha\left({}^C\partial_{x_0}^\alpha u(x_0,x_1,x_2) + {}^C\partial_{x_1}^\alpha v(x_0,x_1,x_2) + {}^C\partial_{x_2}^\alpha w(x_0,x_1,x_2)\right) - u(x_0,x_1,x_2) = 0\\
{}^C\partial_{x_1}^\alpha u(x_0,x_1,x_2) = {}^C\partial_{x_0}^\alpha v(x_0,x_1,x_2),\\
{}^C\partial_{x_2}^\alpha u(x_0,x_1,x_2) = {}^C\partial_{x_0}^\alpha w(x_0,x_1,x_2),\\
{}^C\partial_{x_2}^\alpha v(x_0,x_1,x_2) = {}^C\partial_{x_1}^\alpha w(x_0,x_1,x_2).
\end{array}\right. , \tag{18}$$

where ${}^C\partial_{x_i}^\alpha$, with $i = 0, 1, 2$, represents the Caputo fractional derivative (3) of order $0 < \alpha < 1$ with respect to x_i. We denote its (α, k)-homogeneous polynomial solutions by $R^{(\alpha,k)}$. As it happens in the case of integer derivatives, we can establish a relation between the solutions of (18) and the solutions of (4).

Theorem 4.1 *Let $0 < \alpha \leq 1$. If h is a solution of the fractional Laplace-Beltrami equation (4), then*

$$u(x_0,x_1,x_2) = {}^C\partial_{x_0}^\alpha h(x_0,x_1,x_2), \tag{19}$$

$$v(x_0,x_1,x_2) = {}^C\partial_{x_1}^\alpha h(x_0,x_1,x_2), \tag{20}$$

$$w(x_0,x_1,x_2) = {}^C\partial_{x_2}^\alpha h(x_0,x_1,x_2) \tag{21}$$

is a solution of the hyperbolic fractional Riesz system (18). Conversely, if (u, v, w) is a solution of the hyperbolic fractional Riesz system (18), then there exists locally a function h, satisfying the conditions (19), (20), (21) and being a solution of the fractional Laplace Beltrami equation (4).

Proof: The proof is similar to the correspondent one in classical case (see [11,12]). Nevertheless, to make this paper complete, we present here a sketch of the proof. The proof of the first part follows by straightforward calculations and making use of the Fubini's theorem to verify the last three equations of the system.
We pass now to the proof of the second part. Assume that (u, v, w) is a solution of the hyperbolic fractional Riesz system (18), and define a function by

$$h(x_0,x_1,x_2) = I_{x_0}^\alpha u(x_0,0,0) + I_{x_1}^\alpha v(x_0,x_1,0) + I_{x_2}^\alpha w(x_0,x_1,x_2). \tag{22}$$

It follows by direct calculations that this function gives us a potential, in fact making use of the Fubini's Theorem, Lemma 2.3, and (18) we get

$$\begin{aligned}
{}^C\partial_{x_0}^\alpha h(x_0,x_1,x_2) &= {}^C\partial_{x_0}^\alpha I_{x_0}^\alpha u(x_0,0,0) + {}^C\partial_{x_0}^\alpha I_{x_1}^\alpha v(x_0,x_1,0) + {}^C\partial_{x_0}^\alpha I_{x_2}^\alpha w(x_0,x_1,x_2)\\
&= u(x_0,0,0) + I_{x_1}^\alpha \underbrace{{}^C\partial_{x_0}^\alpha v(x_0,x_1,0)}_{={}^C\partial_{x_1}^\alpha u(x_0,x_1,0)} + I_{x_2}^\alpha \underbrace{{}^C\partial_{x_0}^\alpha w(x_0,x_1,x_2)}_{={}^C\partial_{x_2}^\alpha u(x_0,x_1,x_2)}\\
&= u(x_0,0,0) + I_{x_1}^\alpha\, {}^C\partial_{x_1}^\alpha u(x_0,x_1,0) + I_{x_2}^\alpha\, {}^C\partial_{x_2}^\alpha u(x_0,x_1,x_2)\\
&= w(x_0,0,0) + u(x_0,x_1,0) - u(x_0,0,0) + u(x_0,x_1,x_2) - u(x_0,x_1,0)\\
&= u(x_0,x_1,x_2).
\end{aligned}$$

In a similar way we conclude that

$$ {}^{C}\partial_{x_1}^{\alpha} h(x_0,x_1,x_2) = v(x_0,x_1,x_2), \qquad {}^{C}\partial_{x_2}^{\alpha} h(x_0,x_1,x_2) = w(x_0,x_1,x_2). $$

Taking into account the previous relations it follows that h of the form (22) is such that $\Delta_{lb} h = 0$.

■

Let us denote the set of (α,k)-homogeneous polynomial solutions of the hyperbolic fractional Riesz system by $\mathcal{R}^{(\alpha,k)}$. This means that $(u,v,w) \in \mathcal{R}^{(\alpha,k)}$ if and only of (u,v,w) is a solution of the hyperbolic fractional Riesz system and u, v and w are (α,k)-homogeneous polynomials. Let us define the mapping a mapping $L^\alpha : \mathcal{H}^k_\alpha \to \mathcal{R}^{k-1}_\alpha$ by

$$ L^\alpha h(x_0,x_1,x_2) = \left({}^{C}\partial_{x_0}^{\alpha} h(x_0,x_1,x_2), {}^{C}\partial_{x_1}^{\alpha} h(x_0,x_1,x_2), {}^{C}\partial_{x_2}^{\alpha} h(x_0,x_1,x_2) \right). \tag{23} $$

Since this linear mapping is invertible in $\mathbb{R}^3$, we may deduce the following theorem:

Theorem 4.2 *Let $0 < \alpha \leq 1$. The dimension of the space*

$$ \mathcal{R}^{(\alpha,k)} = \left\{ L^\alpha h(x_0,x_1,x_2) : h \in \mathcal{H}^{(\alpha,k+1)} \right\}. $$

is given by

$$ \dim\left(\mathcal{R}^{(\alpha,k)}\right) = \begin{cases} 1 + \dfrac{k+1}{2}, & (k \text{ odd}) \\ 2 + k, & (k \text{ even}) \end{cases}. $$

Proof: The proof its immediate. In fact, since the linear mapping is an isomorphism we conclude that

$$ \dim\left(\mathcal{R}^{(\alpha,k)}\right) = \dim\left(\mathcal{H}^{(\alpha,k+1)}\right). $$

Making use of Theorem 3.5 we obtain our result.

■

Now we present some examples of some elements of $\mathcal{R}^{(\alpha,k)}$, for some particular values of k and $0 < \alpha < 1$ arbitrary. In all the cases we will apply the mapping L^α to each of the polynomials obtained in Example 3.6.

Example 4.3 *For $k = 1$, applying the mapping L^α to (14), with $0 < \alpha \leq 1$, we obtain the following expressions to $R^{(\alpha,1)} = (u,v,w) \in \mathcal{R}^{(\alpha,1)}$:*

$$ \begin{aligned} u(x_0,x_1,x_2) &= 0; \\ v(x_0,x_1,x_2) &= \varphi(\alpha)\,\beta_{(0,1,0)}; \\ w(x_0,x_1,x_2) &= \varphi(\alpha)\,\beta_{(0,0,1)}. \end{aligned} $$

For $k = 2$, applying the mapping L^α to (15), with $0 < \alpha \leq 1$, we obtain the following expressions to $R^{(\alpha,2)} = (u,v,w) \in \mathcal{R}^{(\alpha,2)}$:

$$ \begin{aligned} u(x_0,x_1,x_2) &= \varphi(2\alpha)\,\beta_{(2,0,0)}\,x_0^\alpha; \\ v(x_0,x_1,x_2) &= \frac{\varphi(2\alpha)\,(\varphi(\alpha)-1)}{\varphi(\alpha)}\,\beta_{(2,0,0)}\,x_1^\alpha - \varphi(2\alpha)\,\beta_{(0,0,2)}\,x_1^\alpha \\ &\quad + \beta_{(0,1,1)}\,\varphi(\alpha)\,x_2^\alpha; \\ w(x_0,x_1,x_2) &= \varphi(\alpha)\,\beta_{(0,1,1)}\,x_1^\alpha + \varphi(2\alpha)\,\beta_{(0,0,2)}\,x_2^\alpha. \end{aligned} $$

For $k = 3$, applying the mapping L^α to (16), with $0 < \alpha \leq 1$, we obtain the following expressions to $R^{(\alpha,3)} = (u,v,w) \in \mathcal{R}^{(\alpha,3)}$:

$$ \begin{aligned} u(x_0,x_1,x_2) &= \varphi(2\alpha)\,\beta_{(2,1,0)}\,x_0^\alpha\,x_1^\alpha + \varphi(2\alpha)\,\beta_{(2,0,1)}\,x_0^\alpha\,x_2^\alpha; \\ v(x_0,x_1,x_2) &= \varphi(\alpha)\,\beta_{(2,1,0)}\,x_0^{2\alpha} + \varphi(\alpha)\,\beta_{(0,1,2)}\,x_2^{2\alpha} \\ &\quad - \left(\varphi(\alpha)\,\beta_{(0,1,2)} + (\varphi(\alpha)-1)\;\beta_{(2,1,0)}\right)\,x_1^{2\alpha} \\ &\quad - \left(\frac{\varphi(3\alpha)\,\varphi(2\alpha)}{\varphi(\alpha)}\,\beta_{(0,0,3)} + \frac{\varphi(2\alpha)\,(\varphi(\alpha)-1)}{\varphi(\alpha)}\,\beta_{(2,0,1)}\right)\,x_1^\alpha\,x_2^\alpha; \\ w(x_0,x_1,x_2) &= \varphi(\alpha)\,\beta_{(2,0,1)}\,x_0^{2\alpha} + \varphi(3\alpha)\,\beta_{(0,0,3)}\,x_2^{2\alpha} + \varphi(2\alpha)\,\beta_{(0,1,2)}\,x_1^\alpha\,x_2^\alpha \\ &\quad - \left(\varphi(3\alpha)\,\beta_{(0,0,3)} + (\varphi(\alpha)-1)\;\beta_{(2,0,1)}\right)\,x_1^{2\alpha}. \end{aligned} $$

For $k=4$, applying the mapping L^α to (17), with $0<\alpha\leq 1$, we obtain the following expressions to $R^{(\alpha,4)}=(u,v,w)\in\mathcal{R}^{(\alpha,4)}$:

$$
\begin{aligned}
u(x_0,x_1,x_2) &= -\frac{\varphi(2\alpha)\,\varphi(\alpha)}{\varphi(3\alpha)-1}\left(\beta_{(2,2,0)}+\beta_{(2,0,2)}\right)x_0^{3\alpha}+\varphi(2\alpha)\,\beta_{(2,2,0)}\,x_0^\alpha\,x_1^{2\alpha}+\varphi(2\alpha)\,\beta_{(2,1,1)}\,x_0^\alpha\,x_1^\alpha\,x_2^\alpha\\
&\quad+\varphi(2\alpha)\,\beta_{(2,0,2)}\,x_0^\alpha\,x_2^{2\alpha};\\
v(x_0,x_1,x_2) &= \varphi(2\alpha)\,\beta_{(2,2,0)}\,x_0^{2\alpha}\,x_1^\alpha+\varphi(\alpha)\,\beta_{(2,1,1)}\,x_0^{2\alpha}\,x_2^\alpha\\
&\quad+\left(\varphi(4\alpha)\,\beta_{(0,0,4)}+\frac{\varphi(2\alpha)\,(\varphi(\alpha)-1)}{\varphi(3\alpha)}\left(\beta_{(2,0,2)}-\beta_{(2,2,0)}\right)\right)x_1^\alpha\,x_2^{2\alpha}\\
&\quad-\left(\varphi(3\alpha)\,\beta_{(0,1,3)}+(\varphi(\alpha)-1)\,\beta_{(2,1,1)}\right)x_1^{2\alpha}\,x_2^\alpha+\varphi(\alpha)\,\beta_{(0,1,3)}\,x_2^{3\alpha}\\
&\quad-\left(\frac{\varphi(4\alpha)\,\varphi(3\alpha)}{\varphi(\alpha)}\,\beta_{(0,0,4)}+\frac{\varphi(2\alpha)\,(\varphi(\alpha)-1)}{\varphi(\alpha)}\,\beta_{(2,0,2)}\right)x_1^\alpha\,x_2^{2\alpha};\\
w(x_0,x_1,x_2) &= \varphi(\alpha)\,\beta_{(2,1,1)}\,x_0^{2\alpha}\,x_1^\alpha+\varphi(2\alpha)\,\beta_{(2,0,2)}\,x_0^{2\alpha}\,x_2^\alpha+\varphi(3\alpha)\,\beta_{(0,1,3)}\,x_1^\alpha\,x_2^{2\alpha}\\
&\quad+\varphi(4\alpha)\,\beta_{(0,0,4)}\,x_2^{3\alpha}-\left(\varphi(\alpha)\,\beta_{(0,1,3)}+\frac{\varphi(\alpha)\,(\varphi(\alpha)-1)}{\varphi(3\alpha)}\,\beta_{(2,1,1)}\right)x_1^{3\alpha}\\
&\quad-\left(\frac{\varphi(4\alpha)\,\varphi(3\alpha)}{\varphi(\alpha)}\,\beta_{(0,0,4)}+\frac{\varphi(2\alpha)\,(\varphi(\alpha)-1)}{\varphi(\alpha)}\,\beta_{(2,0,2)}\right)x_1^{2\alpha}\,x_2^\alpha.
\end{aligned}
$$

Remark 4.4 *If we consider $\alpha=1$ in the polynomials presented in the previous example, we obtain the polynomial solutions of (2), which were introduced by H. Leutwiler. More precisely, we have*

$$
\begin{aligned}
R^{(1,1)}(x_0,x_1,x_2) &= \left(0,\,\beta_{(0,1,0)},\,\beta_{(0,0,1)}\right);\\
R^{(1,2)}(x_0,x_1,x_2) &= \left(2\beta_{(2,0,0)}\,x_0,\,-2\beta_{(0,0,2)}\,x_1+\beta_{(0,1,1)}\,x_2,\,\beta_{(0,1,1)}\,x_1+2\beta_{(0,0,2)}\,x_2\right);\\
R^{(1,3)}(x_0,x_1,x_2) &= \left(2\beta_{(2,1,0)}\,x_0\,x_1+2\beta_{(2,0,1)}\,x_0\,x_2,\beta_{(2,1,0)}\,x_0^2-\beta_{(0,1,2)}\,x_1^2+\beta_{(0,1,2)}\,x_2^2-6\beta_{(0,0,3)}\,x_1\,x_2,\right.\\
&\qquad\left.\beta_{(2,0,1)}\,x_0^2+3\beta_{(0,0,3)}\,x_2^2+2\beta_{(0,1,2)}\,x_1\,x_2-3\beta_{(0,0,3)}\,\beta_{(2,0,1)}\,x_1^2\right);\\
R^{(1,4)}(x_0,x_1,x_2) &= \left(-\left(\beta_{(2,2,0)}+\beta_{(2,0,2)}\right)x_0^3+2\beta_{(2,2,0)}\,x_0\,x_1^2+2\beta_{(2,1,1)}\,x_0\,x_1\,x_2+2\beta_{(2,0,2)}\,x_0\,x_2^2,\right.\\
&\qquad 2\beta_{(2,2,0)}\,x_0^2\,x_1+\beta_{(2,1,1)}\,x_0^2\,x_2+4\beta_{(0,0,4)}\,x_1\,x_2^2-3\beta_{(0,1,3)}\,x_1^2\,x_2\\
&\qquad+\beta_{(0,1,3)}\,x_2^3-12\beta_{(0,0,4)}\,x_1\,x_2^2,\\
&\qquad \beta_{(2,1,1)}\,x_0^2\,x_1+2\beta_{(2,0,2)}\,x_0^2\,x_2+3\beta_{(0,1,3)}\,x_1\,x_2^2+4\beta_{(0,0,4)}\,x_2^3\\
&\qquad\left.-\beta_{(0,1,3)}\,x_1^3-12\beta_{(0,0,4)}\,x_1^2\,x_2\right).
\end{aligned}
$$

To end the paper, the authors will like to point out is that the presented are exactly the same if we consider the Riemann-Liouville fractional derivative of order $0<\alpha<1$, which is defined by (see [4])

$$
\left(\partial_w^\alpha f\right)(w)=\frac{1}{\Gamma(1-\alpha)}\,\frac{\partial}{\partial w}\int_0^w (w-t)^{-\alpha}\,f(t)\;dt,
$$

where $f(w)\in AC^n[0,b]$. In fact, if $f(w)=w^k$ we have that

$$
\left(\partial_w^\alpha f\right)(w)\;=\frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}\,w^{k-\alpha}\;=:\varphi(k)\,w^{k-\alpha},
$$

which coincides with property (6).

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